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A MAXIMUM PRINCIPLE FOR AN ELLIPTIC SYSTEM AND APPLICATIONS TO SEMILINEAR PROBLEMS

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ABSTRACT

The Dirichlet problem in a bounded region for elliptic systems, of the

form

(*) $\qquad -\Delta u = f(x,u) - v, \quad -\Delta v = \delta u - \gamma v$

is studied. For the question of existence of positive solutions the key

ingredient is a maximum principle for a linear elliptic system associated

with (*). A priori bounds for the solutions of (*) are proved under various

types of growth conditions on f. Variational methods are used to establish

the existence of pairs of solutions for (*).

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SIGNIFICANCE AND EXPLANATION

The Dirichlet problem for the semilinear elliptic system

$$(*) \qquad -\Delta u = f(x,u) - v, \quad -\Delta v = \delta u - \gamma v \quad \text{in} \ \Omega,$$

where Ω is a bounded smooth domain in R^N, is studied. Here δ and γ denote positive constants. The solutions (u,v) of $(*)$ represent steady state solutions of reaction diffusion systems of relevance in Biology. The authors consider general classes of nonlinearities f, which are modelled in examples that often appear in the applications. Namely (i) f behaving like $\lambda u - u^3$ where $\lambda > 0$ is some real parameter, and (ii) $f(u) = u(u - a)(1 - u)$, where $0 < a < 1$ is some given real number. A priori bounds for the solutions of $(*)$ are established under various types of growth conditions on f. Then variational methods are used to prove existence of solutions. The linear elliptic system associated with $(*)$ does not fall in the class for which there is a maximum principle available. However, the authors show that in the case of $(*)$ there exists a maximum principle under suitable restrictions on the coefficients. This allows the use of the method of monotone iteration and the establishment of the existence of positive solutions in some cases of interest.

A MAXIMUM PRINCIPLE FOR AN ELLIPTIC SYSTEM
AND APPLICATIONS TO SEMILINEAR PROBLEMS

Djairo G. de Figueiredo* and Enzo Mitidieri**

INTRODUCTION. In this paper we propose to discuss the elliptic system

$$(0.1) \qquad -\Delta u = f(x,u) - v, \quad -\Delta v = \delta u - \gamma v \quad \text{in} \quad \Omega ,$$

where Ω is a bounded smooth domain in R^N, $N > 2$, subject to Dirichlet boundary

conditions $u = v = 0$ on $\partial\Omega$. The solutions (u,v) of this problem represent steady

state solutions of reaction diffusion systems of interest in Biology. Namely systems of

the form

$$(0.2) \qquad u_t = D_1\Delta u + f(u) - v, \quad v_t = D_2\Delta v + \epsilon(u - \gamma v)$$

where D_1, D_2, ϵ and γ are positive constants, and one looks for solutions $u(t,x)$,

$v(t,x)$ defined in $(0,\infty) \times \Omega$, subject to Dirichlet boundary conditions on $(0,\infty) \times \partial\Omega$.

The type of nonlinearities which are of importance in the applications will be described in

the examples I and II below. System (0.2) shows that both species may diffuse. In this

sense it is an extension of the well known FitzHugh Nagumo system, which serves as a model

for nerve conduction, cf. [5] or Hastings [7]. We also mention Koga-Kuramoto [10], where

the complete system (0.2) appears and steady state solutions are discussed. There is an

extensive bibliography in this subject. We mention three additional papers, which are more

closely related to the investigation presented here, namely Rothe-de Mottoni [13], Rothe

[14] and Lazer-McKenna [11].

In the applications the constants γ and δ, which appear in system (0.1), are taken

to be _positive_. So we shall make this assumption throughout this paper. It follows then

that the second equation in (0.1) can be solved for v in terms of u. Let us denote by

B its solution operator under Dirichlet boundary conditions. That is, given u we

define Bu as the solution of the problem $-\Delta v + \gamma v = \delta u$ in Ω, $v = 0$ on $\partial\Omega$. Thus our

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problem becomes the one of finding u such that

(0.3) $\qquad -\Delta u + Bu = f(x,u)$ in Ω, $u = 0$ on $\partial\Omega$.

We observe that the left side of (0.3) contains a local (differential) operator $-\Delta$, and a nonlocal (integral) operator B. This fact gives rising to quite interesting questions. It is essential at the outset to understand the operator $-\Delta + B$. In Section 1 we study its spectral properties and establish a maximum principle for solutions of linear equations like

(0.4) $\qquad -\Delta u + Bu - \lambda u = g(x)$ in Ω, $u = 0$ on $\partial\Omega$,

where the real parameter λ is restricted to certain ranges depending on γ, δ and the region Ω. In Section 2 we establish a priori bounds for solutions of (0.3) under the main assumption that the nonlinearity f at ∞ is below the smallest eigenvalue of the operator $-\Delta + B$; this assumption will be stated precisely as condition (f2) and it characterizes a class of systems which are here called sublinear. The two examples below, which were treated by previous authors [9], [11], [13] and [14], are included in the classes studied in the present paper. Their results are therefore sharpened as far as ranges of the parameters involved and signs of the solutions.

Example I. $f(u) = \lambda u - g(u)$, where λ is a real parameter larger than the first eigenvalue of the operator $-\Delta + B$, and g is a function behaving like u^3, but not necessarily odd. Cf. [11], [13], [14].

Example II. $f(u) = u(u - a)(1 - u)$, where a is such that $0 < a < 1/2$. This is the sort of nonlinearity arising in the FitzHugh-Nagumo equations, [5], [9].

The a priori bounds obtained in Section 2 will be needed in an essential way to perform appropriate truncations of the nonlinearity f, so the problem could be treated by variational methods. This will be done in Section 5.

In Section 3 we discuss a class of systems whose model nonlinearity is the one given by Example I. Using the results of Section 1 we are able to establish the existence of a positive and a negative solution. This result complements a previous one by Lazer and McKenna [11], who proved the existence of two nontrivial solutions by topological degree

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arguments. Their method however does not yield the signs of the solutions obtained. The maximum principle for equations like (0.4) comes very useful in this respect.

In Section 4 we sketch a result on the existence of positive solutions for a superlinear elliptic system. Results similar to the ones known for the scalar case hold true in view of the aforementioned maximum principle. The question of the a priori bounds for positive solutions of superlinear elliptic systems may be a hard one. If the growth of the nonlinearity at $+\infty$ is at most like $(N + 1)/(N - 1)$, for $N > 3$, then the results of Brézis-Turner [2] extend readily. The range $[(N + 1)/(N - 1),(N + 2)/(N - 2)]$ poses serious difficulties. The methods used in de Figueiredo-Lions-Nussbaum [3] to treat the scalar case rely on the results of Gidas-Ni-Nirenberg [6], which are not available as yet for the type of systems studied here. We remark that Troy [15] has extended some of the results in [6] to systems. However Troy's systems do not include the ones we are concerned with. Also in Section 4 we prove a nonexistence result basing it on our extension to systems of the well known Pohozaev's identity.

In Section 5 we consider a class of systems whose model nonlinearity is the one given in Example II. Using the Mountain Pass Theorem of Ambrosetti-Rabinowitz [1] we establish Theorem 5.1 on the existence of two nontrivial solutions for such systems, extending a previous result of Klaasen-Mitidieri [9]. This result shows clearly the relevance of the volume of Ω and of the parameters γ and δ on the existence questions. It also exhibits the importance of a large positive parameter λ on the existence of two positive solutions for the system

$$-\Delta u = \lambda f(x,u) - v \quad -\Delta v = \delta u - \gamma v, \quad \text{in} \quad \Omega$$

subject to Dirichlet boundary conditions, and the nonlinearity f is of the type given by Example II. This relates to the scalar case studied in Rabinowitz [12].

The contents of this paper is as follows:

1. The operator $-\Delta + B$

2. A priori bounds for solutions of sublinear elliptic systems

3. Existence of positive solutions

4. Remarks on a superlinear system

5. Existence of two nontrivial solutions for a class of sublinear systems

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1. **THE OPERATOR** $-\Delta + B$. Consider the linear Dirichlet problem

(1.1) $$-\Delta v + \gamma v = \delta u \quad \text{in } \Omega, \quad v = 0 \quad \text{on } \partial\Omega ,$$

where $\Omega \subset R^N$ is a bounded and smooth domain, γ and δ are positive constants. Let us denote by B its solution operator: $v = Bu$. It is well known that

$$B : L^2(\Omega) \to H^2(\Omega) \cap H_0^1(\Omega); \quad B : L^p(\Omega) \to W^{2,p}(\Omega); \quad B : C^\alpha(\bar{\Omega}) \to C^{2+\alpha}(\bar{\Omega}) .$$

Let us define the operator

$$T \equiv -\Delta + B : L^2(\Omega) \to L^2(\Omega), \quad \text{with} \quad D(T) = H^2(\Omega) \cap H_0^1(\Omega) .$$

Clearly T is symmetric, that is, $(Tu_1, u_2) = (u_1, Tu_2)$ for all $u_1, u_2 \in D(T)$, where $(,)$ denotes the L^2 inner-product. Using the L^2 regularity theory one can prove that T is a closed operator. Let us denote by $0 < \lambda_1 < \lambda_2 < \lambda_3 < \dots$ the eigenvalues of $-\Delta$ under Dirichlet boundary conditions, and by ϕ_k the corresponding eigenfunctions. Then it is easily verified that

(1.2) $$\hat{\lambda}_k = \lambda_k + \frac{\delta}{\gamma + \lambda_k}, \quad k = 1, 2, \dots$$

are eigenvalues of T. Moreover the same ϕ_k's defined above are their corresponding eigenfunctions. Since (ϕ_k) is a complete orthonormal set in L^2, it is readily shown that the $\hat{\lambda}_k$'s are the _only_ eigenvalues of T. We shall prove in the sequel that in fact the **spectrum** $\sigma(T)$ of T **consists precisely of these eigenvalues.** For each λ in the resolvent set $\rho(T)$ of T, let us denote by $T_\lambda = (T - \lambda I)^{-1}$ its corresponding resolvent operator.

LEMMA 1.1. (A representation formula of the resolvent operator for some values of λ). Suppose that the real numbers a and b satisfy the following conditions

(1.3) $$a > -\lambda_1, \quad \gamma + b > -\lambda_1, \quad b \neq 0 , \quad \underline{\text{and}}$$

(1.4) $$b\gamma + \delta = ab .$$

Then $\lambda = -a - b$ is in the resolvent set $\rho(T)$ and

(1.5) $$T_\lambda = [1 - b(\gamma + b - \Delta)^{-1}](a - \Delta)^{-1} .$$

Proof. With $\lambda = -a - b$, one can write

$$T - \lambda I = (a - \Delta) + b\left[\frac{b\gamma + \delta}{b} - \Delta\right](\gamma - \Delta)^{-1} .$$

Using condition (1.4) above one obtains

$$T - \lambda I = (a - \Delta)[1 + b(\gamma - \Delta)^{-1}] = (a - \Delta)(\gamma - \Delta)^{-1}(\gamma + b - \Delta) .$$

Finally using condition (1.3) it follows that

$$T_\lambda = (\gamma + b - \Delta)^{-1}(\gamma - \Delta)(a - \Delta)^{-1} ,$$

which readily gives (1.5).   □

Remark 1.1. A calculation shows that λ, taken in the ranges indicated below, are representable as $\lambda = -a - b$, with a and b satisfying (1.3) and (1.4):

(i)   All $\lambda < -\gamma - 2\sqrt{\delta}$. These λ's correspond to $b > 0$.

(ii) If $\gamma + \lambda_1 > \sqrt{\delta}$, there are some additional values of λ. Namely $2\sqrt{\delta} - \gamma < \lambda < \lambda_1 + \frac{\delta}{\gamma + \lambda_1}$. These λ's correspond to b negative in the range $-\lambda_1 - \gamma < b < -\frac{\delta}{\gamma + \lambda_1}$.

Remark 1.2. (Monotonicity of the sequence $\hat{\lambda}_k$). We observe that $\gamma + \lambda_1 > \sqrt{\delta}$ implies that $\hat{\lambda}_1 < \hat{\lambda}_2 < \hat{\lambda}_3 < \ldots$. Of course one does not have in general such a monotonicity of the eigenvalues $\hat{\lambda}_k$. Clearly $\gamma + \lambda_1 > \sqrt{\delta}$ is not a necessary condition, since it in fact implies the stronger statement that the function $s \longmapsto s + \frac{\delta}{\gamma + s}$ is monotonically increasing in the whole halfline $[\lambda_1, \infty)$. A necessary and sufficient condition for this monotonicity involves also the second eigenvalue λ_2, namely $\delta < (\gamma + \lambda_1)(\gamma + \lambda_2)$.

Corollary 1.2. (Compactness of T_λ). For all $\lambda \in \rho(T)$, the resolvent operator T_λ is compact.

Proof. For any $\lambda, \mu \in \rho(T)$ one has the resolvent equation

$$T_\mu - T_\lambda = (\mu - \lambda)T_\mu T_\lambda .$$

So if T_λ is compact for some λ, then it is compact for all λ's in the resolvent set. By the previous lemma T_λ is compact for $\lambda < -\gamma - 2\sqrt{\delta}$.   □

The following result is an immediate consequence of Lemma 1.1 and Remark 1.1 above.

Corollary 1.3. (**Positiveness of** T_λ **for some values of** λ). **If** $\gamma + \lambda_1 > \sqrt{\delta}$, **then** T_λ **is positive for all** $2\sqrt{\delta} - \gamma < \lambda < \hat{\lambda}_1$.

Remark 1.3. The positiveness of T_λ is a <u>maximum principle</u> for the equation

$$-\Delta v + Bv - \lambda v = u \quad \text{in } \Omega, \quad v = 0 \quad \text{on } \partial\Omega .$$

It says that if $u \in L^2$ and $u > 0$ a.e., then $v > 0$ a.e. In fact, it follows from the representation formula (1.5) that a <u>strong maximum principle</u> holds. Namely, if $u \in C^0(\Omega)$ and $u > 0$ in Ω, then $v > 0$ in Ω and the outward normal derivative $\frac{\partial v}{\partial \nu} < 0$. [Recall that Ω is being assumed to be smooth. So the interior sphere condition is satisfied].

Remark 1.4. If $\gamma > 2\sqrt{\delta}$, then the condition $\gamma + \lambda_1 > \sqrt{\delta}$ is automatically satisfied, and Corollary 1.3 says that in this case T_λ is positive for λ in an interval which contains 0. In general one cannot expect that T_0 be positive. Indeed, if $\gamma = \delta = 1$, then Corollary 1.3 says that T_λ is positive for $1 < \lambda < \hat{\lambda}_1$.

Proposition 1.4. **The spectrum** $\sigma(T)$ **of** T **consists of precisely the eigenvalues** $\hat{\lambda}_k$.

Proof. We have seen above that the point spectrum $P\sigma(T) = \{\hat{\lambda}_k : k = 1, 2, \ldots\}$. Let $\lambda \notin P\sigma(T)$. Then $T - \lambda I$ is one-to-one. If we show that $T - \lambda I$ is onto, it follows by the Closed Graph Theorem that $\lambda \in \rho(T)$. Thus we claim that equation $Tu - \lambda u = v$ has a solution u for each given $v \in L^2$. Taking $\mu \in \rho(T)$ we see that this equation is equivalent to $Tu - \mu u = (\lambda - \mu)u + v$, or

$$(1.6) \qquad\qquad u = (\lambda - \mu)T_\mu u + T_\mu v .$$

By Fredholm alternative (1.6) is solvable iff the homogeneous equation $u = (\lambda - \mu)T_\mu u$ has only the solution $u = 0$. But this is actually the case, since this homogeneous equation is equivalent to $Tu = \lambda u$. Recall that $\lambda \notin P\sigma(T)$. $\qquad\qquad\square$

Remark 1.5. The above proposition follows also from general results in Functional

Analysis. Namely, T being a self-adjoint operator it follows that its residual spectrum

$R\sigma(T)$ is empty. Next, since $-\Delta - \lambda$ is Fredholm for every $\lambda \in C$, it follows that

$-\Delta + B - \lambda$ is also Fredholm for all $\lambda \in C$. Consequently the continuous spectrum $C\sigma(T)$

is also empty.

Remark 1.6. (An useful inequality). Let $\tilde{\lambda}$ denote the smallest of the eigenvalues $\hat{\lambda}_k$.

We have seen above that $\tilde{\lambda} = \hat{\lambda}_1$ if $\gamma + \lambda_1 > \sqrt{\delta}$. We assert that

$$(1.7) \qquad\qquad (Tu,u) > \tilde{\lambda}|u|^2_{L^2} , \qquad \forall u \in D(T) .$$

Indeed, since (ϕ_k) is a complete orthonormal set in L^2, we can write $u = \sum \alpha_k \phi_k$ where

$\alpha_k = (u,\phi_k)$. So

$$(Tu,u) = \sum \alpha_k (Tu,\phi_k) = \sum \alpha_k (u,T\phi_k) = \sum \alpha_k^2 \hat{\lambda}_k ,$$

from which the claim follows. A similar argument shows that

$$(1.8) \qquad\qquad \int |\nabla u|^2 + (Bu,u) > \tilde{\lambda}|u|^2_{L^2} , \qquad \forall u \in H_0^1 .$$

Remark 1.7. (Uncoupling of systems and maximum principles). The usual maximum principle

for systems, as well as the maximum principle proved here, seems to be related with the

possibility of uncoupling the elliptic system. To make precise our observation, let us

look at the linear elliptic system

$$(1.9) \qquad\qquad -\Delta u = au + bv + f(x)$$

$$-\Delta v = cu + dv + g(x)$$

subject to Dirichlet boundary conditions: $u = v = 0$ on $\partial\Omega$, where Ω is some bounded

domain in R^N, and a,b,c and d are real constants. Suppose that $b \neq 0$ and $c \neq 0$;

otherwise the problem trivializes. The uncoupling of system (1.9) is possible if the

matrix of the coefficients

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

has two distinct eigenvalues, μ_1 and μ_2. Such a condition is equivalent to

$$(1.10) \qquad\qquad (a - d)^2 + 4bc > 0 .$$

Of course this is the case if b and c both have the same sign. However to infer the signs of u and v from the signs of the corresponding functions in the uncoupled system one needs that both b and c be positive. This gives the usual maximum principle for systems. On the other hand if b and c have opposite signs the uncoupling is still possible provided a and d "compensate" for the negativeness of bc. Through some calculations one can prove the following result, which essentially gives our maximum principle.

Proposition 1.5 In addition to (1.10) assume that $bc < 0$, $c(a - d) > 0$, $\mu_1 < \hat{\lambda}_1$ and $\mu_2 < \hat{\lambda}_1$. Then if $f > 0$, $g > 0$ and $cf > (a - \mu_1)g$, it follows that the solutions and v of (1.9) are positive in Ω.

2. A PRIORI BOUNDS FOR SOLUTIONS OF SUBLINEAR ELLIPTIC SYSTEMS.

Let us consider the elliptic system

(2.1) $\qquad -\Delta u = f(x,u) - v, \quad -\Delta v = \delta u - \gamma v \quad \text{in } \Omega$,

where Ω is a bounded smooth domain in R^N, subject to Dirichlet boundary conditions. We always assume that γ and δ are positive constants. The nonlinearity f is subject to the following conditions.

(f1) $f : \bar{\Omega} \times R \to R$ is locally Lipschitzian,

(f2) $\lim\sup\limits_{|s| \to \infty} \dfrac{f(x,s)}{s} < \tilde{\lambda}$ (uniformly in Ω), where $\tilde{\lambda}$ denotes the smallest eigenvalue of the operator $-\Delta + B$ studied in Section 1. Condition (f2) characterizes system (2.1) as being sublinear.

Examples. 1) $f(u) = \lambda u - h(u)u$, where h is a c^1 function such that $h(0) = 0$, $h'(s)s > 0$ for all $s \neq 0$ and $\lim\inf\limits_{s \to \pm\infty} h(s) > \lambda$, (for instance $h(s) = s^2$). This is the case considered in [11] and [14].

2) $f(u) = u(u - a)(1 - u)$, where $0 < a < 1$. This is the type of nonlinearity that appears in the FitzHugh-Nagumo equations. Cf. [5], [9].

Remark 2.1. By a solution of (2.1) we mean a classical solution. That is, a pair of functions (u,v) which are in $C^2(\Omega) \cap C^0(\bar{\Omega})$ and which are 0 on $\partial\Omega$. We observe that if $u,v \in H^1_0(\Omega) \cap C^0(\bar{\Omega})$ satisfy (2.1) in the distribution sense, then by a bootstrap argument it follows that $u,v \in C^{2,\alpha}(\bar{\Omega})$. We remark that in general one cannot drop the hypothesis that u and v are in $C^0(\bar{\Omega})$ in order to be able to bootstrap. However, this would be possible provided one assumes some growth condition on f.

In order to obtain the a priori bound for the solutions of (2.1) we shall assume either one of the conditions below.

(f3) $\displaystyle\lim_{|s|\to\infty} \frac{f(x,s)}{|s|^p} = 0$, where $1 < p < \dfrac{N+2}{N-2}$, if $N > 3$, and $1 < p < \infty$, if $N = 2$

(f4) $\displaystyle\limsup_{|s|\to\infty} \frac{f(x,s)}{s} < -\frac{\delta}{\gamma}$,

where the limits are uniform in Ω.

Remark 2.2. In the scalar case (i.e. $-\Delta u = f(x,u)$) condition (f4) corresponds to $f(x,s) < 0$ for $s > \beta > 0$ and $f(x,s) > 0$ for $s < -\beta$, where β is some real number.

Proposition 2.1. Under hypotheses (f1), (f2) and (f3), the solutions of (2.1) are a priori bounded in L^∞.

Proof. It follows from (f2) that there exist $0 < \mu < \tilde{\lambda}$ and $M > 0$ such that

(2.2) $f(x,s) \leq \mu s + M$, for $0 \leq s < \infty$; $f(x,s) \geq \mu s - M$ for $-\infty < s < 0$.

The second equation in (2.1) can be solved for v in terms of u. And in this way system (2.1) is equivalent to the equation

(2.3) $-\Delta u + Bu = f(x,u)$,

using the notation of Section 1. So we need only to prove bounds on u. The corresponding bounds on v are obtained immediately from the second equation in (2.1). Multiplying (2.3) by u, integrating by parts and using (1.8) we obtain

(2.4) $\tilde{\lambda} \int u^2 \leq \int |\nabla u|^2 + \int (Bu)u = \int f(x,u)u$

Next we estimate the last term in (2.4) using (2.2)

(2.5) $\int f(x,u)u \leq \mu \int u^2 + M \int |u|$

which implies $\int u^2 < C$. (We shall use the same C to denote different constants).
Using (2.4) and (2.5) again and recalling that B is a bounded linear operator in L^2, we
conclude that $\int |\nabla u|^2 < C$. It follows from (f3) that given $\varepsilon > 0$ there exists $C_\varepsilon > 0$
such that

$$|f(x,s)| < \varepsilon |s|^p + C_\varepsilon .$$

Finally using this inequality and invoking L^p estimates and the Sobolev imbedding
theorem, we conclude that there exist a constant $\hat{C}$ such that $|u|_{L^\infty} < \hat{C}$.　　　□

Remark 2.3. We emphasize that the dependence of $\hat{C}$ on f is through the constants
μ, M and C_ε. So if we change f for $|s| > \hat{C}$ maintaining μ, M and C_ε, the new
equation (2.3) with this modified f has the same solutions of the original equation
(2.3). This fact will be used in Section 5.

The following result was proved by Rothe [14] and Lazer and McKenna [11] under less
general hypotheses on f. The main idea in the proof below is taken from those papers.

Proposition 2.2. Under hypotheses (f1), (f2) and (f4), the solutions of (2.1) are a priori
bounded in L^∞.

Proof. (i) We first claim that for $u \in C^0(\bar{\Omega})$, with $u = 0$ on $\partial\Omega$, one has
(2.6) $$\frac{\delta}{\gamma} \min u < (Bu)(x) < \frac{\delta}{\gamma} \max u, \quad x \in \Omega .$$
Indeed we know that $v = Bu$ satisfies the equation
(2.7) $$v = \frac{1}{\gamma} \Delta v + \frac{\delta}{\gamma} u$$
Let us prove the first inequality in (2.6). If $v \geqslant 0$ that inequality is trivially
true. So let us assume that for $x_1 \in \Omega$ we have $v(x_1) = \min v < 0$. Then $\Delta v(x_1) \geqslant 0$
and (2.7) implies that $v(x_1) \geqslant \frac{\delta}{\gamma} u(x_1)$, from which the first inequality in (2.6) follows
readily. In a similar way we prove the second inequality in (2.6).

(ii) It follows from (f4) that there exist positive constants k and m such that
(2.8) $$\frac{f(x,s)}{s} < -k < -\frac{\delta}{\gamma} , \quad |s| > m .$$

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We claim that $\|u\|_{L^\infty} < m$ for all solutions u of (2.1). Indeed, suppose by contradiction that $\|u\|_{L^\infty} = M > m$ for some solution u. It follows from (2.6), using the first equation in (2.1) that

(2.9) $$\frac{\delta}{\gamma} \min u < \Delta u + f(x,u) < \frac{\delta}{\gamma} \max u .$$

If there is $x_0 \in \Omega$ such that $u(x_0) = M$ we obtain from (2.9) and (2.8) that

$$-\frac{\delta}{\gamma} M < f(x_0, u(x_0)) < -ku(x_0) = -kM ,$$

which is impossible. In a similar way we arrive to a contradiction if $u(x) = -M$ for some $x \in \Omega$. $\qquad\square$

Next we discuss the question of bounds for positive solutions of the system (2.1). As remarked before we need only to obtain bounds on u, and then corresponding bounds on v follow readily.

Proposition 2.3. In addition to (f1) assume the following condition

(f5) there exists a constant $m > 0$ such that $f(x,s) = 0$ for $s > m$.

Then all nonnegative solutions u of (2.3) are bounded above by m.

Proof. Given a solution u of (2.1) define the function ω as $\omega(x) = u(x) - m$ for $u(x) > m$ and $\omega(x) = 0$ for $u(x) < m$. Such a ω belongs to $H_0^1(\Omega)$. So it follows from (2.3) that

(2.10) $$\int |\nabla\omega|^2 + (Bu,\omega) = \int f(x,u)\omega .$$

In view of (f5) and the fact that $(Bu)(x) > 0$ for $x \in \Omega$, we conclude from (2.10) that $\int |\nabla\omega|^2 = 0$, which implies $\omega = 0$. $\qquad\square$

Remark 2.4. This proposition will be used as follows. Suppose that the function f is such that there is an $m > 0$ for which $f(x,m) = 0$. Then we consider system (2.1) with f replaced by a new function $\tilde{f}$ defined as f for $s < m$ and as 0 for $s > m$. If for this new system we could find a nonnegative solution u, then by the proposition above such a u would be indeed a solution of the original system.

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Remark 2.5. Now if $f(x,s) = f(s)$ satisfies (f2), $f(0) > 0$ and $f(s) > 0$ for $0 < s < \varepsilon$, then either there is an $m > 0$ such that $f(m) = 0$ or f satisfies (f3). In the first case we treat the problem as in the previous remark. In the second case we proceed as in Proposition 2.1 and obtain an a priori bound on positive solutions.

Remark 2.6. Similar statements can be made for nonpositive solutions u.

Remark 2.7. A sufficient condition for all (eventual) nontrivial solutions of (2.1) to be positive. Assume that $\gamma + \lambda_1 > \sqrt{\delta}$ and that $f(x,u) > \alpha u$ for all u, where $-\gamma + 2\sqrt{\delta} < \alpha < \hat{\lambda}_1$. Then the nontrivial solutions u of (2.1) are positive in Ω. From (2.3) we obtain $-\Delta u + Bu > \alpha u$, and the result follows readily by Corollary 1.3.

Remark 2.8. The previous condition applied to Example 2 gives interesting conclusions. Indeed, we can in this case compute explicitly the value of m in (2.8). Then truncate f outside $|s| > m$ in such a way that the new f has derivative equals to $-a$ for $|s| > m$. By Proposition 2.2 the solutions of (2.1) with this new f are the same as the solutions of the original equation. Moreover, from the way the truncation is done, it follows (by a straightforward calculation) that now $f(u) > -au$, (where we are denoting also by f the truncated function) provided $\delta/\gamma < a$. So the previous sufficient condition applies. Summarizing, the solutions of (2.1), in the case of Example 2, are positive if

(2.11)
$$\frac{\delta}{\gamma} < a < \gamma - 2\sqrt{\delta}$$

Observe that, if (2.11) is assumed, then the condition $\gamma + \lambda_1 > \sqrt{\delta}$ is automatically satisfied, cf. Remark 1.4. We remark that no solution of (2.3) in this example can be nonpositive (i.e. $u < 0$ in Ω). In fact the solutions in general change sign.

3. EXISTENCE OF POSITIVE SOLUTIONS.

We consider again system (2.1) of the previous section or its equivalent expression in the form of equation (2.3). In this section we examine the question of existence of a positive solution under an additional condition on the nonlinearity f at 0. In order to simplify the presentation in the sequel we suppose that f does not depend on x. The case when f depends also on x can also be treated by the method used here; under appropriate conditions on f similar results may be obtained. So we assume the condition next.

(f6) $\lim\inf\limits_{s\to 0} \dfrac{f(s)}{s} > \hat{\lambda}_1$

Examples. Condition (f6) is satisfied, for instance, if (i) $f(0) > 0$, or (ii) $f(s)$ is C^1 and $f'(0) > \hat{\lambda}_1$. A special case of (ii) was considered in [11].

Theorem 3.1. Assume that $\gamma + \lambda_1 > \sqrt{\delta}$. In addition to conditions (f1), (f2) and (f6), suppose that f is C^1 for $s > 0$ and

(3.1) $\inf\{f'(s) : 0 < s < \beta\} > -\gamma + 2\sqrt{\delta}$,

where $\beta < +\infty$ is the first positive zero of $f(s)$. Then equation (2.3) has a positive solution u, or equivalently, system (2.1) has a pair (u,v) of positive solutions.

Remark 3.1 The hypothesis $\gamma + \lambda_1 > \sqrt{\delta}$ in Theorem 3.1 implies that $\tilde{\lambda} = \hat{\lambda}_1$. Recall also that under this hypothesis $-\gamma + 2\sqrt{\delta} < \hat{\lambda}_1$, and so we can make use of Corollary 1.3. The condition on the differentiability of f can be relaxed and in consequence (3.1) has to be replaced by an appropriate one-sided Lipschitz condition.

Proof of Theorem 3.1. (i) It follows from (f6) that there exist $\nu > \hat{\lambda}_1$ and $s_0 > 0$ such that $f(s) > \nu s$ for $0 < s < s_0$. Thus $\varepsilon\phi_1$ is a subsolution of (2.3) for all ε such that $0 < \varepsilon < \varepsilon_0 = s_0/\max \phi_1$.

(ii) If $\beta < \infty$ then $\omega(x) \equiv \beta$ in Ω is a supersolution of (2.3). If $\beta = +\infty$ we construct a supersolution ω for (2.3) as follows. It follows from (f2) that there exist $-\gamma + 2\sqrt{\delta} < \mu < \hat{\lambda}_1$ and $C > 0$ such that $f(s) < \mu s + C$. We then take ω as the solution

of $-\Delta\omega + B\omega = \mu\omega + C$ in Ω, $\omega = 0$ on $\partial\Omega$. In view of Corollary 1.3 $\omega > 0$ in Ω and $\varepsilon > 0$ can be chosen in such a way that $\varepsilon\phi_1 < \omega$ in Ω.

(iii) So (2.3) possesses an ordered pair of a sub- and a supersolution. Now in order to apply the method of monotone iteration, it is still required that (a) $T_\lambda = (-\Delta + B - \lambda I)^{-1}$ be a positive operator for some real number λ, and (b) the function $s \longrightarrow f(s) - \lambda s$, for the same λ, be nondecreasing in the interval $[0, \max \omega]$. These two requirements are accomplished if one chooses $\lambda = -\gamma + 2\sqrt{\delta}$. Indeed, (a) then follows by Corollary 1.3 and (b) follows from (3.1). Therefore the method of monotone iteration can be applied and one obtains a solution of (2.3) in the interval $[\varepsilon\phi_1, \omega]$.

Remark 3.2. It should be remarked that besides (f2) no growth condition is required on f.

Remark 3.3. A statement similar to Theorem 3.1 holds true for the existence of negative solutions of (2.1). In this case, condition (3.1) is replaced by

$(3.1)^-$ $\qquad\qquad\qquad \inf\{f'(s) : \beta' < s < 0\} > -\gamma + 2\sqrt{\delta}$

where $-\infty < \beta' < 0$ is the first negative zero of $f(s)$. In order to prove such a result we can reduce it to the situation of Theorem 3.1 by the substitution $z = -u$.

Example. $f(u) = au - u^3$ with $a > 0$. In this case $\beta = \sqrt{a}$, and $\min\{f'(s) : 0 < u < \beta\} = -2a$. So conditions (f6) and (3.1) are satisfied if $\hat{\lambda}_1 < a < \gamma/2 - \sqrt{\delta}$. We then see that in this example there are values of a for which (2.3) has a positive solution provided

(3.2) $\qquad\qquad\qquad\qquad \hat{\lambda}_1 < \gamma/2 - \sqrt{\delta}$.

Clearly this is the case for instance if γ is large. This is also the case if $\gamma > (\sqrt{3} + 1)\sqrt{\delta}$ and Ω is a sufficiently large ball. Indeed, for large balls λ_1 is essentially zero and this last inequality implies readily condition (3.2). Clearly in this example there is also a negative solution, namely $-u$, where u is the positive solution.

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Comparison with the results of Lazer-McKenna. In [11] the following system is studied

(3.3) $$-k\Delta u = \lambda u - h(u)u - v, \quad -\Delta v + v = u \quad \text{in} \quad \Omega$$

subject to Dirichlet boundary conditions. Under certain conditions on k, λ and h it is proved that

(3.4) $$-k\Delta u + (1 - \Delta)^{-1}u = \lambda u - h(u)u ,$$

which is an equivalent form of (3.3), has exactly three solutions. In [11] a topological degree argument is used, which does not give the sign of the two nontrivial solutions. Under essentially the same hypotheses, our Theorem 3.1 says that one of these solutions is positive and the other is negative. Our precise result is the following. We state only the one corresponding to the existence of a positive solution. A similar one can be drawn for the existence of a negative solution.

Corollary 3.2. Under the assumptions below, equation (3.4) has a positive solution:

(3.5) $$1 + \lambda_1 > 1/\sqrt{k}$$

(3.6) $$h \in C^1(R,R), \quad h(0) = 0, \quad h'(s)s > 0, \quad \forall s \neq 0 .$$

(3.7) $$k\lambda_1 + \frac{1}{1 + \lambda_1} < \lambda$$

(3.8) $$\sup\{h'(s)s + h(s) : 0 < s < \beta\} < \lambda + k - 2\sqrt{k} ,$$

where β **is the only positive solution of** $h(s) = \lambda$. **(Observe that** β **could be** $+\infty$**).**

Remark 3.4. If $h'(s)$ is nondecreasing then $\beta < \infty$ and (3.8) simplifies to $\beta h'(\beta) < k - 2\sqrt{k}$. So positive solutions of (3.4) exist if the **diffusion rate** k **is large**.

4. REMARKS ON A SUPERLINEAR SYSTEM. Consider the elliptic system

$$(4.1) \qquad\qquad -\Delta u = f(u) - v, \quad -\Delta v = \delta u - \gamma v \quad \text{in } \Omega ,$$

subject to Dirichlet boundary conditions, with $\gamma, \delta > 0$ and $-\gamma + 2\sqrt{\delta} < 0$. Assume the following conditions on the nonlinearity f

(f1)' $f : R^+ \to R^+$ locally Lipschitzian,

(f7) $\lim\limits_{s \to +\infty} \inf \dfrac{f(s)}{s} > \hat{\lambda}_1$

(f8) $\lim\limits_{s \to 0} \sup \dfrac{f(s)}{s} < \hat{\lambda}_1$

(f9) $\lim\limits_{s \to +\infty} \dfrac{f(s)}{s^\sigma} = 0$ where $1 < \sigma < (N + 1)/(N - 1)$, if $N \geqslant 3$ and $1 < \sigma < \infty$,

$$\text{if } N = 2.$$

As seen in the previous section, (4.1) is equivalent to

$$(4.2) \qquad\qquad -\Delta u + Bu = f(u) .$$

Under the hypotheses above we may proceed as in the scalar ease (cf. Brézis-Turner [2]) and we prove that (4.2) has a positive solution. Condition (f9) is used to get a priori bounds for the positive solutions of (4.2). We do not know how to proceed in order to obtain such bounds in the case when $(N + 1)/(N - 1) < \sigma < (N + 2)/(N - 2)$ and $N \geqslant 3$. The results of [3] for the scalar ease are not immediately extended to this case. For that purpose, the first step would be to see how the results of Gidas-Ni-Nirenberg [6] look (if at all!) in this case. We remark that the extension obtained by Troy [15] does not cover the type of systems studied in this paper.

Remark 4.1. The condition $-\gamma + 2\sqrt{\delta} < 0$ is used in order to guarantee that the operator $T_0 \equiv (-\Delta + B)^{-1}$ is positive. If this condition is not satisfied, but one has $\gamma + \lambda_1 > \sqrt{\delta}$, everything still works provided $\hat{\lambda}_1$ in the right sides of assumptions (f7) and (f8) is replaced by $\hat{\lambda}_1 - \gamma + 2\sqrt{\delta}$.

Nonexistence of positive solutions in the case when $f(u) = u^p$, for $p \geqslant (N + 2)/(N - 2)$ and $N \geqslant 3$. As in the scalar case this is proved using an identity of the Pohozaev type.

The function $f(u) = u^p$ for $u > 0$ is extended as $f(u) = 0$ for $u < 0$. Then it follows from Remark 2.7 that all eventual solutions u and v of (4.1) are positive in Ω, provided we assume that $-\gamma + 2\sqrt{\delta} < 0$. Consequently the nonexistence of nontrivial solutions for system (4.1) (in star-shaped domains Ω) with such an f follows readily from the two lemmas below.

Lemma 4.1. Let u and v be solutions of (4.1). Then the following identity holds

(4.3) $2N \int F(u) - (N - 2) \int uf(u) - 2 \int uv - \frac{2}{\delta} \int |\nabla v|^2 = \oint (x \cdot \nu)\left[|\nabla u|^2 - \frac{1}{\delta} |\nabla v|^2 \right]$

where $F(s) = \int_0^s f$ and $\int$ denotes (volume) integral over Ω and $\oint$ (surface) integral over $\partial\Omega$. Here ν denotes the outward unit normal.

Lemma 4.2. Let u and v be solutions of (4.1). Assume that $-\gamma + 2\sqrt{\delta} < 0$. Then $u - (1/\sqrt{\delta})v$ is positive in Ω and

$$\frac{\partial u}{\partial \nu} < \frac{1}{\sqrt{\delta}} \frac{\partial v}{\partial \nu} < 0 \quad \text{on} \quad \partial\Omega .$$

To conclude this section we prove the two lemmas above.

Proof of Lemma 4.1. First we use the general form of Pohozaev's identity for solutions of the $-\Delta u = g(x,u)$ in Ω and $u = 0$ on $\partial\Omega$; see [3]. This identity will be applied separately to the first and second equations in (4.1). Observe that for the first equation, $g(x,s) = f(s) - v(x)$, and for the second equation, $g(x,s) = \delta u(x) - \gamma s$. Then we obtain the following two identities

(4.4) $2N \int [F(u) - uv] - 2 \int (x \cdot \nabla v)u - (N - 2) \int [f(u) - v]u = \oint (x \cdot \nu)|\nabla u|^2$

(4.5) $2N \int [\delta uv - \frac{1}{2} \gamma v^2] + 2\delta \int (x \cdot \nabla u)v - (N - 2) \int [\delta u - \gamma v]v = \oint (x \cdot \nu)|\nabla v|^2$

(If one prefers to ignore [3], identities (4.4) and (4.5) may be obtained in the standard way Pohozaev's identities are proved. Use the multiplier $x \cdot \nabla u$ in the first equation of (4.1) and $x \cdot \nabla v$ in the second). It follows from the divergence theorem that

(4.6)
$$\int (x \cdot \nabla v)u + \int (x \cdot \nabla u)v = -N \int uv$$

Next dividing (4.5) through by δ, subtracting the result from (4.4) and using (4.6) we obtain

(4.7)
$$2N \int F(u) - (N-2) \int uf(u) - 4 \int uv + \frac{2\gamma}{\delta} \int v^2 = \oint (x \cdot \nu)[|\nabla u|^2 - \frac{1}{\delta} |\nabla v|^2] .$$

Now it follows from the second equation in (4.1) that

(4.8)
$$\int |\nabla v|^2 = \delta \int uv - \gamma \int v^2 .$$

Taking (4.8) into (4.7) we obtain the identity (4.3). □

Proof of Lemma 4.2. It follows from $-\gamma + 2\sqrt{\delta} < 0$ that there eixsts a real number k such that $\sqrt{\delta} < k < \gamma - \sqrt{\delta}$. Using (4.1) it is easy to check that

$$(-\Delta + k)\left(u - \frac{1}{\sqrt{\delta}} v\right) > 0 \quad \text{in} \quad \Omega$$

from which the assertion of the lemma follows. Observe that we know that all (eventual) solutions of (4.1) would be positive. □

5. EXISTENCE OF TWO NONTRIVIAL SOLUTIONS FOR A CLASS OF SUBLINEAR SYSTEMS.

Let us once more consider system (2.1) under conditions (f1), (f2), (f3) or (f4). As in previous sections we discuss, instead of system (2.1), its equivalent form given by equation (2.3). In this section we propose to treat the question of existence of solutions of (2.3) by a variational argument. So we look for the critical points of the functional

(5.1)
$$\Phi(u) = \frac{1}{2} \int |\nabla u|^2 + \frac{1}{2} (Bu, u) - \int F(x, u)$$

where $F(x, s) = \int_0^s f(x, \xi) d\xi$. Although this functional is well defined in H_0^1 if we assume (f3), this is not the case if (f4) is assumed instead. Observe that both (f2) and (f4) restrict f only in one direction. So some truncation has to be done. The existence of a priori bounds on the solutions of (2.3) in either case ((f3) or (f4) assumed), as proved in Section 2, allows us to truncate the nonlinearity f in such a way that the functional Φ

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is well defined in H_0^1 and it is bounded from below. Indeed, in case (f3) is assumed we choose an appropriate $\tilde{c} > \hat{c}$ and do this truncation for $|s| > \tilde{c}$ (see Proposition 2.1 and Remark 2.3) preserving μ, M and C_ϵ and in such a way that $\lim\limits_{|s| \to \infty} \dfrac{f(x,s)}{s} = \ell$ where $0 < \ell < \tilde{\lambda}$. In case we assume (f4) the truncation is done for $|s| > m$ (see Proposition 2.2), and in such a way that $\lim\limits_{|s| \to \infty} \dfrac{f(x,s)}{s} = -k$, where the constant k is given in (2.8). The truncation so done has the very essential feature that the new equation (2.3) with this truncated function has the same solutions as the solutions of the original equation (2.3).

It is immediate to see that $\Phi : H_0^1(\Omega) \to R$ is C^1 and

$$(5.2) \qquad (\Phi'(u),\omega)_{H^1} = \int \nabla u \cdot \nabla \omega + \int (Bu)\omega - \int f(x,u)\omega .$$

So the critical points of Φ are the H_0^1 solutions of (2.3). By a bootstrap argument it follows that these solutions are in fact in $C^{2,\alpha}(\bar{\Omega})$.

Lemma 5.1. The functional Φ defined above satisfies the Palais Smale condition.

Proof. (i) In view of Poincaré's inequality we may consider H_0^1 endowed with the inner product $(u,\omega)_{H^1} = \int \nabla u \cdot \nabla \omega$. It is well known that the nonlinear operator $\tilde{f} : H_0^1 \to H_0^1$ defined by $(\tilde{f}(u),\omega)_{H^1} = \int f(x,u)\omega$, $\forall \omega \in H_0^1$, is compact. (Recall that f has linear growth in view of the truncation). On the other hand the (linear) operator $\tilde{B} : H_0^1 \to H_0^1$ defined by $(\tilde{B}u,\omega)_{H^1} = \int (Bu)\omega$ is also compact. This follows readily from the compact imbedding of H_0^1 in L^2. Consequently $\Phi' = I + \tilde{B} - \tilde{f}$, that is, Φ' is of the form identity + compact operator. Thus to prove the Palais Smale condition it is enough to show that any sequence $(u_n \in H_0^1$ such that $|\Phi(u_n)| < C$ and $\Phi'(u_n) \to 0$ in H_0^1 possesses a subsequence (denoted again by u_n) such that $\|u_n\|_{H^1} < C$.

(ii) It follows from $\Phi'(u_n) \to 0$ that given $\epsilon_n \downarrow 0$ there exists a subsequence of (u_n) (denoted again by u_n) such that

$$(5.3) \qquad |\int \nabla u_n \cdot \nabla \omega + \int (Bu_n)\omega - \int f(x,u_n)\omega| < \epsilon_n \|\omega\|_{H^1} .$$

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Now using (5.3) with $\omega = u_n$ and estimating with the help of (1.8) we get

$$(5.4) \qquad \tilde{\lambda} \int |u_n|^2 < \int f(x,u_n)u_n + \epsilon_n |u_n|_{H^1}$$

From the properties of the truncated f we obtain from (5.4)

$$(5.5) \qquad \int u_n^2 < C + C\epsilon_n |u_n|_{H^1}$$

Next from $|\Phi(u_n)| < C$ we infer that

$$(5.6) \qquad \int |\nabla u_n|^2 < \int (Bu_n)u_n + 2 \int |F(x,u_n)| + C$$

and finally using the properties of the truncated f we obtain from (5.6) and (5.5) that

$\int |\nabla u_n|^2 < C + C\epsilon_n |u_n|_{H^1}$ which proves that $|u_n|_{H^1} < C$. $\qquad\qquad\square$

Remark 5.1. It follows immediately from the previous remarks that system (2.1) has <u>at least one solution</u> under hypotheses (f1), (f2), and (f3) or (f4). Indeed, since Φ is C^1 functional, bounded below and satisfying the Palais Smale condition, it follows that it has a global minimum u_1, $\Phi(u_1) = \inf\{\Phi(u) : u \in H_0^1\}$. One cannot expect in general the existence of more solutions. Indeed if $f(u) = \lambda u$ with $\lambda < \tilde{\lambda}$, equation (2.3) in this case has only the trivial solution! So some additional assumption is necessary.

Now we treat a problem which is superlinear at 0, in the sense that the condition below holds

(f10) f is differentiable at 0, $f(x,0) = 0$, and $f'(x,0) < \tilde{\lambda}$.

Example 2 in Section 2 satisfies condition (f10).

Theorem 5.2. <u>Assume conditions (f1), (f2), (f3) or (f4), and (f10). In addition suppose that there exists</u> $\xi > 0$ <u>such that</u>

$$(5.7) \qquad F(\xi) > F(s) \qquad \forall \; 0 < s < \xi .$$

$$(5.8) \qquad \frac{2F(\xi)}{\xi^2} > \min\left\{ \frac{1}{R^2} \frac{(1+t)^2}{t^2} \frac{(1+t)^N - 1}{2 - (1+t)^N} + \frac{\delta}{\gamma} \frac{(1+t)^N}{2 - (1+t)^N} : 0 < t < 2^{1/N} - 1 \right\}$$

where R denotes the radius of the largest ball contained in Ω. Then equation (2.3) has at least two nontrivial solutions.

Remark 5.2. Condition (5.8) is the analogue of a condition introduced by one of the authors (D.G.F.) in [4] for the scalar case. We remark that if there is a $\xi > 0$ such that $F(\xi) > 0$ then condition is satisfied for example if Ω is a large ball and δ is very small. The special case of Example 2 was studied by Klaasen and Mitidieri [9]. Condition (5.8) follows readily from their conditions: (i) Ω to be a large ball, and (ii) $\dfrac{\gamma}{\delta} > \dfrac{q}{2a^2 - 5a + 2}$.

Proof. It suffices to prove that there exists $\widetilde{u} \in H_0^1$ such that $\phi(\widetilde{u}) < 0$. Once this is done we see that the global minimum u_1 of ϕ is a nontrivial solution since $\phi(u_1) = \inf \phi < 0$. The second solution is obtained immediately by an application of the Mountain Pass Theorem of Ambrosetti-Rabinowitz [1], since 0 is a strict local minimum in view of assumption (f10). In order to see that there are points in H_0^1 where the functional ϕ is negative we consider the functions u_t below. We may assume that the ball centered at 0 with radius R is contained in Ω, where R is the radius of the largest ball contained in Ω. Defining

$$u_t(x) = \begin{cases} \xi, & \text{if } |x| \leq R/(1 + t) \\[2mm] \xi\left[1 - \dfrac{1 + t}{tR}\left(|x| - \dfrac{R}{1 + t}\right)\right], & \text{if } \dfrac{R}{1 + t} < |x| \leq R \\[2mm] 0, & \text{if } x \in \Omega \backslash B_R(0) \end{cases}$$

the result follows by a calculation from conditions (5.7) and (5.8). $\square$

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The Dirichlet problem in a bounded region for elliptic systems of the form

$$-\Delta u = f(x,u) - v, \quad -\Delta v = \delta u - \gamma v$$

is studied. For the question of existence of positive solutions the key ingredient is a maximum principle for a linear elliptic system associated with (*). A priori bounds for the solutions of (*) are proved under various types of growth conditions on f. Variational methods are used to establish the existence of pairs of solutions for (*).

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